■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 1
FALL 1991

Editor's Welcome

This newsletter will be a crazy-quilt collection of riddles, brain-teasers, math projects, paper-folding exercises, puzzles, games, magic, etc.

Sometimes you might think, "What does this problem have to do with mathematics?" But there is much more to mathematics and mathematical thinking than doing arithmetic. Any activity involving mental visualization and manipulation of objects can be considered either mathematics or as training for mathematics.

Now let's get down to business!

Hints, answers, or solutions to problems will appear in the next issue.

1. First a Few Riddles

just to get our minds in motion

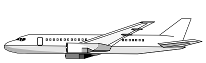
What did one arithmetic book say to the other?

Boy, do you think you have problems!

Why are mosquitoes good at arithmetic?

*They **add** to your misery, they **subtract** from your pleasure, they **divide** your attention, and they **multiply** like crazy!*

An airplane crashes exactly on the bor-



der between Mexico and the United States. Do you bury the survivors in Mexico or the U.S.?

You don't bury survivors, silly!

2. It Was a Dark and Stormy Night

a long tale

It was a dark and stormy night, and the girls were gathered around the campfire. One camper asked the leader to tell them a story. The camp leader said:

> It was a dark and stormy night, and the girls were gathered around the campfire. One camper asked the leader to tell them a story. The camp leader said:
>
> > It was a dark and stormy night, and the girls were gathered around the campfire. One camper asked the leader to tell them a story. The camp leader said:
> >
> > > *It was a dark and stormy night, and the girls were gathered around the campfire. One camper asked the leader to tell them a story. The camp leader said: "No way!"*

Comments

This story illustrates the notion of **infinity**—that things can go on forever. If we could make the printed letters smaller and smaller it might even be possible to let the story go on forever, yet still have it fit on this one page. However, the print would get so small that you could not read all of it!

The concept of infinity should be familiar to you because you know the example 1,2,3,4,... that goes on forever. No matter how big a number is, you can always make a bigger one by adding 1. What is not so familiar is the idea that you can add infinitely many things together and still get something **finite.** The standard example of this is the infinite sum: 1 + a half + a quarter + an eighth + a sixteenth + etc. = 2

3. The Clock in the Mirror

a little brain-teaser

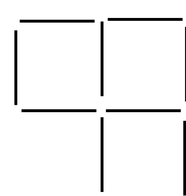
You see a clock reflected in a mirror, and it appears to say the time is 2:30. What time is it really?

4. A Toothpick Puzzle

an exercise in perception

A popular kind of puzzle involves arranging toothpicks.

Take away 2 toothpicks from the set-up below and be left with exactly 2 squares!

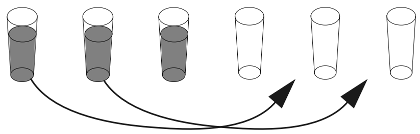


In the next issue we will present a harder one of this type. You can make up your own problems like this and try them out on your friends.

5. The Full and Empty Glasses

a tricky puzzle---so watch out!

Take six water glasses and fill 3 with water, leaving the remaining 3 empty. Line them up as follows:



Notice that by moving the first two glasses as shown by the arrows you can put the glasses into the order **full-empty-full-empty-full-empty** (namely, *alternating* full and empty) A little tougher job *if you rise to the challenge* is to do the same thing by moving *only one glass!*

6. Turning a T-shirt inside-out

an exercise in "practical" topology

There is a branch of higher mathematics called **topology** (pronounced top-PAH-low-gee) that studies smooth changes of shape. A **topologist** thinks that a coffee mug is the same thing as a doughnut. (Even though they *taste* totally different!) The idea is, if you made the coffee mug out of rubber, and worked hard enough, maybe you could make it look like a doughnut. Try to visualize it!

This is a good thing to try with a friend or member of your family. It seems like it should be impossible, so you can get their interest by saying something like, "I bet you I can take my T-shirt off and put it back on so that it is inside-out, even though my hands are tied together!"

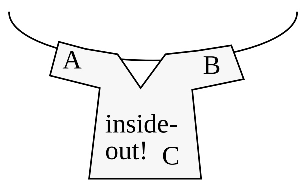
The trick involves taking your shirt off and putting it back on, so before you get started, you might want to put on **two** T-shirts. Girls might prefer to wear swim suits underneath.

You need a length of rope or a narrow belt. Have someone tie the two ends of the rope to each of your wrists, leaving at least 2 feet of rope between your wrists. (This is so you will be able to use both hands. If you have an assistant who knows the trick, you could just have your hands tied together, and the assistant could manipulate the T-shirt.)

Once you are ready, with your shirt on and your wrists tied with rope, tell your audience what you are going to do: "I am going to take off my shirt and then put it back on so that it is inside-out, and when I finish, the rope will be between my wrists just as it is now."

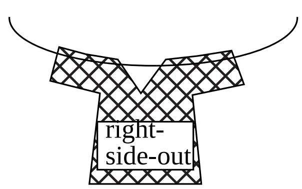
First step: take off your shirt!

Cross your arms, grasp your shirt on the bottom sides, pull it off over your head, bring it down in front of you, and spread your arms apart! Display the shirt on the rope. It is now inside-out, but if you were



to put it back on now, it would be right-side-out again, and you would have proved nothing!

Second step: turn the shirt right-side-out while it is on the rope (think "clothesline").



It is not easy to explain how to do this. It involves pushing B and C through the hole in the sleeve A.

Third step: put your shirt back on.

This is more challenging than taking the shirt off, even though it is just the reverse process! What you do is stick your arms back into the shirt through the sleeves, then out at bottom, raise your arms above your head, wiggle around, and *hope for the best!* (Use your hands if you have to.)

7. The Tired Mother

this could happen!

A tired mother goes to sleep at 8:00 P.M. after setting her alarm clock for 9 o'clock the next morning. When the alarm went off, she still was very tired. How much sleep did she get?

8. Secret Messages

based upon changing the letters

Two common ways to make secret messages (the art of **cryptography** (crip-TOG-grah-fee)) are called **substitution** and **transposition.** In the first method the letters stay where they are, but get changed. For example, you could change A to B, B to C,..., Y to Z, and Z to A. If the secret message is: NOW IS A GOOD TIME TO EAT A SNACK, then after you change each letter you get:OPX JT B HPPE UJNF UP FBU B TOBDL. Given such a mess, you can get back to the original secret message by changing B to A, C to B, D to C,...,Z to Y, and A to Z.

See if you can figure out what the following message is:

EP ZPV MJLF NS TDIJMEFS CFUUFS XJUI PS XJUIPVU IJT NVTUBDIF?

Of course, if you use such a method to disguise your messages, and somebody gets a copy of your message, that *enemy* could guess what you did and figure out the message for himself. So you probably want to do something a little more complicated, involving you and your friends knowing a **secret key** that allows you and your friends to read the messages, but keeps your enemies out.

HOPEWELL ELEMENTARY SCHOOL

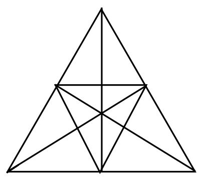
MATH MINDS IN MOTION

NUMBER 2
FALL 1991

1. Triangles to Drive You Crazy

a geometry counting problem

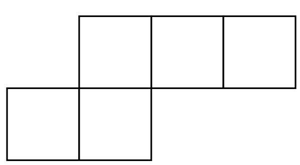
Find as many triangles as you can hidden in the following diagram:



2. A Harder Toothpick Puzzle

in case you thought the first one was too easy

In the diagram below are some number of toothpicks arranged to make 5 small squares:

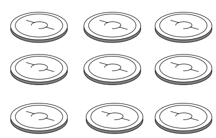


How many toothpicks are there?

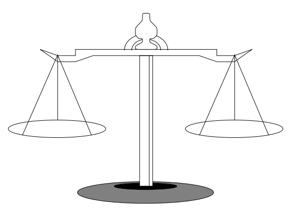
How can you move exactly 2 toothpicks to other spots in the picture and produce a diagram containing exactly 4 small squares? (No doubling up of toothpicks allowed!)

3. The 9 Coins and the Balance Scales

looking for the fake coin



Someone gives you 9 coins that look the same. Eight of them are good **gold coins**, but one of them is a fake gold coin—it is just coated with gold to make it look real. You know that the fake coin is lighter than a good coin, and the good coins all weigh the same. You have a **balance.**



Depending on which way it tips, you can tell whether one collection of coins is heavier, lighter, or the same weight as another collection of coins. With only 2 "balancings" you can be sure to find the fake coin. How do you do this?

4. Fibonacci Sequences Simplified

how to generate a lot of numbers without much work

Everyone is familiar with the sequence 1,2,3,4,5,...—it is used for counting things. The rule for making the **next** guy in the sequence is just "add 1 to the current guy!" What could be simpler?

The **Fibonacci** (FIB-oh-NOTCH-ee) sequence is made by adding the two most recent numbers together. Start with 1,1. Then 1+1=2, 1+2=3, 2+3=5, 3+5=8, 5+8=13, and so on. One bad thing is that the numbers get bigger and bigger, and pretty soon we grow tired of all the work. Instead, why don't we just keep track of one digit: throw away any carries! Then we get the sequence 1,1,2,3,5,8,3, 1,4,5,9,4,3,7,0,7,7,... How long is it until this sequence starts repeating itself?

5. Another Secret Message

this time we rearrange the letters

If you have a group of friends, and want to send secret messages to each other, first you should decide on a **secret key** which will be used to encode your messages. For the current method we need to chose a *phony* telephone number that contains the numbers 0,1,2,...,9 in some rearranged order, with no numbers appearing more than once. For example, we take our secret key to be the number:

(609) 472-8135

Your and your friends would pick such a number, and write it down in your diaries or address books. Anybody looking in your books would think it was someone's telephone number, and might not be suspicious that it was your secret message key number!

Make up a secret message that is exactly 100 letters long. (You might have to break off in the middle of a word.) Here is our example secret message, where we have already removed all the spaces:

ONWEDNESDAYATTHREEO-
CLOCCKMEETBYTHEGYM-
DOORFORFURTHERINSTRUCT
IONSABOUTWHATTOBRING-
TOCAMPTHISWEEKENDSTOP

Notice that we use the word STOP in place of a period, to mark the end of a sentence. (Also notice the misspelling OCLOCCK. Sorry! Nobody's perfect.)

We are going to use a **transposition** system. Namely, we will scramble up the order of the letters in a way determined by our secret key number. The same scrambling method can be used to unscramble the scrambled message and get the original message back.

First make an 11 by 11 sheet of squares:

6	10	9	4	7	2	8	1	3	5	
							c	c		6
							o	o		10
							l	l		9
							u	u		4
							m	m		7
r	o	w		2			n	n		2
										8
r	o	w		1			1	3		1
										3
										5

Put the secret key numbers across the top, one per column. Also put the same numbers on the right side, one per row. Write 10 instead of 0. Take the message and start writing its letters into the column labeled 1, from top to bottom, then go on to the column labeled 2, and so on, finishing with the column labeled 10. Your should get the following filled-up array of letters:

6	10	9	4	7	2	8	1	3	5	
R	E	O	Y	O	Y	A	O	L	R	6
I	E	C	T	N	A	T	N	O	F	10
N	K	A	H	S	T	T	W	C	O	9
S	E	M	E	A	T	O	E	C	R	4
T	N	P	G	B	H	B	D	K	F	7
R	D	T	Y	O	R	R	N	M	U	2
U	S	H	M	U	E	I	E	E	R	8
C	T	I	D	T	E	N	S	E	T	1
T	O	S	O	W	O	G	D	T	H	3
I	P	W	O	H	C	T	A	B	E	5

Now, you read left-to-right out each of the rows labeled 1,2,3,... to produce the following **cipher**:

CTIDTENSETRDTYORRNMU-
TOSOWOGDTHSEMEATO-
ECRIPWOHCTABEREOYOYAOL
RTNPGBHBDKFUSH-
MUEIEERNKAHSTTWCOIECT-
NATNOF

Now see if you can decode the following message, where I've put a space every 10 letters, so you can keep the columns on cut:

ISEUNNNEIO TTTMBVUAES
GTVMATCFNP AIELFSERGN
IEEBCREOPT HRRRTRCTIE
SXAEIEQELE DYSNOEERST
LSPBFYSENO EYEOOTIHAR

6. Santa and his Socks

The Dirichlet Pigeon-Hole Principle—a famous but easy math fact

Some facts in mathematics are so important that they are named after their inventors. The German mathematician Lejeune Dirichlet (pronounced deer-EESH-lay), who lived from 1805 until 1859, made much use of the following simple principle:

> If you have more items than boxes, and you put each item into one of the boxes, then when you finish there will be at least one box with more than one item in it!

For example, if there are only 20 kinds of candy-bar and 21 children each buy one, then at least 2 children must have bought the same kind of bar.

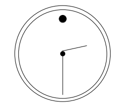
Here is a problem. Suppose Santa Claus has a drawer full of socks. Twenty are red, ten are white, and sixteen are green, but they are all mixed together and not paired up. He gets up in the middle of the night, and so as not to wake up his wife, Mrs. Claus, he does not turn on the light in the bedroom, but just grabs a bunch of socks at random from his drawer. When he gets to his workshop, he will turn on the lights, see if he got at least two socks of the same color, and put them on. How many socks must he grab in order to make sure that he will have a matching pair of socks?

Answers and Solutions

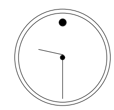
for issue number 1

2. And the story continues:"...it was a dark and stormy night..."

3. If you see 2:30 in the mirror, what you see is:

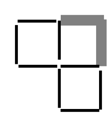


So the real clock looks like:



Therefore, it is really 9:30. Without thinking carefully about this, many adults say 10:30 or 11:30!

4. In order to get 2 squares, remove the two toothpicks that are shaded



5. Pick up the middle of the 3 full glasses, pour its water into the middle glass of the empty ones, then put the glass back where it started!

6. You have to do it in order to believe it!

7. Only 1 hour. Her old alarm clock was a mechanical one that did not know the difference between A.M. and P.M. She went to bed at 8 and the alarm went off one hour later, at 9.

8. The secret message is

DO YOU LIKE MR SCHILDER BETTER WITH OR WITHOUT HIS MUSTACHE?

■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 3
FALL 1991

1. Things with Peculiar Prices

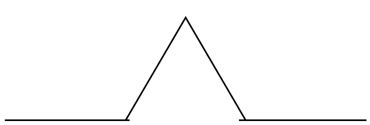
a brain-teaser (trick problem)

A boy went to the hardware store and bought some things for his mother. When he got back home, his mother asked him, “How much did 157 cost?” and he answered, “Three dollars.” Then she asked, “How much did 86 cost?” and he answered, “Two dollars.” Finally she asked, “How much did 1 cost?” and he answered, “One dollar.” What was the boy buying?

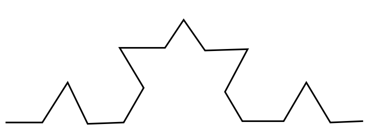
2. A Fractal Doodle

snowflake curves

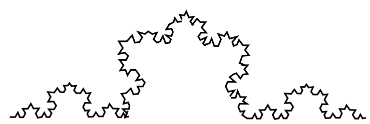
A fractal is a curve that stays bumpy no matter how closely you look at it. When I was a kid, I used to doodle sometimes by starting with a line segment, dividing it into three equal parts, then replacing the middle third by 2 equal sides, obtaining:



Then, I would divide each of those 4 segments into 3 and replace their middle thirds with “bumps” similarly:



Keeping this up, I eventually got something like:



which is called a *snowflake curve*. Try it for yourself! I recommend that you use a sharp pencil, and erase the middle thirds.

Each time that you take out the middle thirds and replace them, the curve gets more jagged, and it gets longer. In fact, if you could keep going forever, you’d produce a curve that has **infinite** length!

3. Cooking Steak on the Bar-B-Que

an “operations research” problem

Dad is cooking 3 steaks on a small outdoor charcoal grill that only has room to cook 2 steaks at a time. Each steak needs to be cooked for 5 minutes on each side. His plan for cooking the 3 steaks will take 20 minutes. First he will take 10 minutes to cook both sides of 2 steaks, and then he will take 10 minutes to cook the third steak. His daughter, Sonya, tells him that she has a plan that takes only 15 minutes. Can you figure out what she has in mind?

4. Boiling Numbers Down to a Single Digit

the famous “digital sum”

Here is a simple way to check your work for arithmetic problems. First, here is an example of how to “boil down” a number to 1 digit. Start with a big number, say 137865. Add up its digits: 1+3+7+8+6+5=30. Then, add up 30’s digits: 3+0=3. We say that 3 is the **digital sum** of 137865.

Suppose that you have done the following arithmetic problem: 567+328, gotten the answer 905, and want to double check your work. (Fat chance!) First, boil down 567 to 18 and then to 9, boil down 328 to 13 to 4. Then, instead of adding 567 to 328, add the digital sums 9+4=13 and boil down to 4. If we did the problem correctly, then we should get 4 also when we boil down the answer 905. Let’s check: 905 goes to 9+0+5=14, which goes to 1+4=5. Aha! we must have made a mistake. In fact, 567+328=895. See, 895 boils to 8+9+5=22, which boils to 2+2=4 as required.

Of course, this is only a “check”—it is not a guarantee that you have the correct answer. There are tons of numbers that boil down to the same digit.

This also works for checking multiplication.

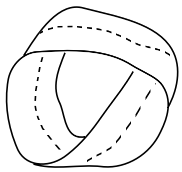
The boiling down process is sometimes called the “method of casting out nines.” The digital sum is what you get if you keep subtracting 9 from your number until you are down to one digit. Also, you don’t have to bother adding 9’s when you see them. For example, to boil down the number 99991999991999991999, first you “cast out the nines” getting 111, which you then boil down to 3.

Boil down the number 123456789.

5. The One-Sided Piece of Paper

the "Moebius strip" math experiment

Take a strip of paper about an inch wide and about 12 inches long and tape the ends together to form a loop, after twisting the strip by one "half-twist." You will get something that looks like the picture below. This is a Moebius strip, and is an



example of a "one-sided" surface. Imagine a tiny boy and a tiny girl standing on the strip, thinking that the strip is a highway. The girl goes out to explore, and walks all the way around the strip, without going over the edge. When she comes back to the boy, she will be on the "other side" of the paper. After one more loop she will be back to where she started, standing next to the boy. A mathematician says that there is only one side, because you can get everywhere on the surface without crossing an edge.

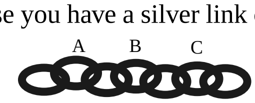
Draw a dotted center line on the Moebius strip highway, and then cut along the center line, all the way around. Before cutting, try to imagine what will happen.

You will get a new strip. Is it one-sided or two-sided now? How many half-twists are in it? Draw a new center line for this new strip. Cut along the center line. What do you get?

6. The seven link silver chain

making change out of a good piece of jewelry

Suppose you have a silver link chain and



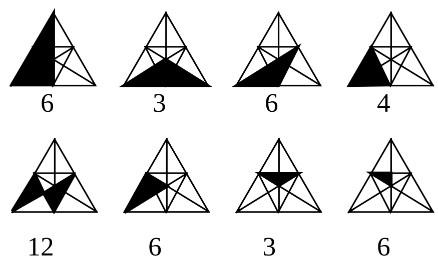
each link is worth $1. If you were to cut the links marked A, B, and C, you could take the chain apart and have 7 separate pieces. Then you could make up any of these amounts of money: $1, $2, $3, $4, $5, $6, $7. By cutting only **one** link, can you make all those same amounts of money? If so, how do you do it?

Answers and Solutions

for issue number 2

1. Triangles to drive you crazy

I was almost driven crazy myself! I found 47. In order to do this, I had to be very careful, making a picture of each type of triangle, and counting how many there were of that type. In addition to the 1 largest triangle, here is what I found:

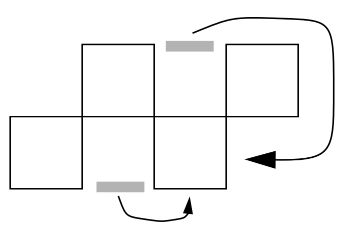


Below each picture is the number of triangles I found of that type. Then, I just added up all those numbers: 1+6+3+6+4+12+6+3+6=47.

2. A harder toothpick puzzle.

There are 16 toothpicks. (Ha! You probably were expecting this to be a trick question. If so, I tricked you!)

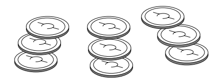
By moving 2 toothpicks you can obtain the following diagram where I have



marked the old positions of the 2 toothpicks.

3. The 9 coins and the balance scales.

Divide the 9 coins into 3 groups of 3 coins. The light coin is in one of these groups. Put 2 of the groups on the balance scales, one group on each side. If the scales don't tip, then those groups



weigh the same, so the bad coin must be in the third group of 3. If the scales tip one way, then the bad coin is in the lighter group. In any case, we have narrowed the bad coin down to a group of 3. Now put one of those coins on one side of the scale and another on the other side. If they weigh the same, the third coin is the bad one. If one weighs less than the other, then that is the bad one. Whew!

Do you see how you could use the same idea to find the light coin in 27 coins by using the balance only 3 times?

If you don't know whether the bad coin is heavy or light, the problem is more difficult. Can you find the bad coin in a group of 10 by using the balance only 3 times? It is also possible to do 11 or 12, but it gets trickier and trickier.

4. Fibonacci simplified

1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,
7,7,4,1,5,6,1,7,8,5,3,8,1,9,0
9,9,8,7,5,2,7,9,6,5,1,6,7,3,0,
3,3,6,9,5,4,9,3,2,5,7,2,9,1,0,
1,1,2,3,5,8,3,1,4,5,9,4,3,7,0

The sequence repeats itself every 60 steps.

5. Another secret message.

THERE ARE FORTY SEVEN TRIANGLES IN PROBLEM NUMBER ONE. STOP. THE LAST DIGIT OF FIBONACCI SEQUENCE REPEATS EVERY SIXTY STEps (I didn't have room for the last 2 letters in the message square.)

6. Santa and his Socks. Santa needs to take only 4 socks. To apply the Dirichlet pigeon-hole principle, think of Santa having 3 boxes, labeled Red, White, and Green. He takes his 4 socks and sorts them each into the correct box. Since he put 4 things into only 3 boxes, one of the boxes has to have at least 2 socks in it, and he finds a matching pair!

■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 4
FALL 1991

1. Mrs. Trip's Triplets

first it was socks—now it is hats!

Mrs. Trip has identical triplets that she always dresses alike. In a drawer she has 3 red , 3 pink, 3 yellow, and 3 green baby hats, all mixed together. If she picks hats out at random, without looking, hoping to get three that are the same color, what is the largest number of hats she might take out *without* getting 3 that match?

2. Three Boys with Painted Faces

logical thinking at play

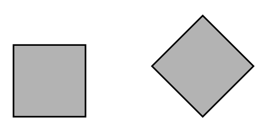
Three boys have a picnic pig-out of fried chicken in the park, and eat so much that they all lie down and fall asleep. Some girls decide to play a trick on them. They sneak up on the boys, paint green blotches on their foreheads, and then go off and hide. When the boys wake up (and who do you suppose woke them up?) they each look at the other two and start laughing. (Of course, each one thinks he is OK, and the other two have silly blotches on their faces.) After a short time, one of them stops laughing, because he realizes that he must have been painted, too. How does he figure that out?

Answer. (Hold on to your thinking caps!) Let's first give the boys names, so we can talk about them. The smartest one is Adam, and the other two are Bart and Clem. Adam thinks to himself, "Those guys look pretty silly. Laugh, laugh. Bart is laughing, so he must be laughing at Clem. And Clem must be laughing at Bart. But, wouldn't Bart be thinking as follows, "Hey, why is Clem laughing? I must have a blotch, too!" and then stop laughing? But, Bart continues to laugh. Why doesn't he figure out that he has a blotch? Can he be that stupid? No, way. He should have figured it out by now. Oh, oh, I must have a blotch myself."

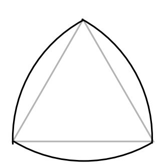
3. A Curve of Constant Width

that is not a circle

Everybody knows that a wheel works well only if it is shaped like a circle. Other shapes would make the car bounce up and down. One property that the circle has is that of "constant width". That means that no matter how you turn it, it will always have the same width. (In fact, no matter how you turn a circle, it will be exactly the same in all respects!) Let's check to see if a square has "constant width." First put it down flat and then tip it up straight. See that it is taller (and



wider) when it is tipped up. If you had a bunch of logs with the square shape (as seen from the ends) it would be hard to roll something resting on top of them, but for normal, circular logs, it would be easy to roll. What is a little surprising is that there are other shapes besides the circle that have constant width, and would be OK for making "rolling logs." Here is one:



Each side is a part of a circle that has its center at the opposite corner. Suppose that you put a square exactly around it. Then, as the rounded triangle turns around inside its box, it will always be touching all 4 sides of the square.

4. The Worm in the Encyclopedia

this kind of book worm is bad for knowledge

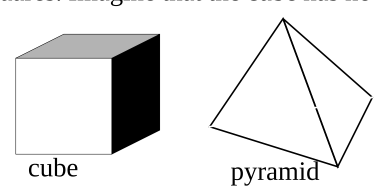
If a worm eats from page 1 of volume A through to the last page of volume D, and each book is 1 inch thick, how far does the worm go? (Ignore the thickness of the book covers.)

Hint. The answer is **not** 4 inches. Go to a bookshelf and see where the first page of a book is (if the book isn't upside-down).

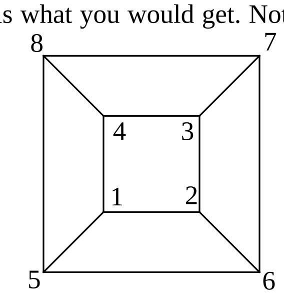
5. Flattened Polyhedra

making flat maps of some common objects

Polyhedra are 3-dimensional shapes made up of lines, points, and flat sides. Mathematicians call points **vertices**, lines **edges,** and sides **faces**. Probably the most common one is the cube, which is made up of 6 faces, 12 edges, and 8 vertices. The faces in this case are all squares. Imagine that the cube has no in-

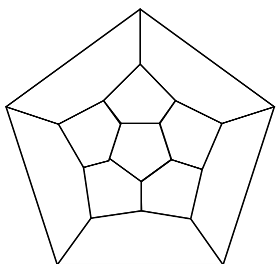


side and it is stretchable. Remove one side and then mentally flatten the rest. Here is what you would get. Notice that



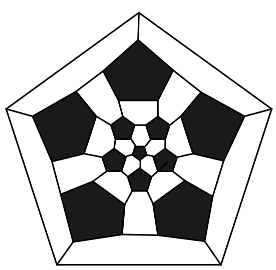
I have marked the 8 vertices. From this flat map it is easy to check that there are 5 faces and 12 edges. (Remember that we took one face away, so we could flatten the cube—there used to be 6 faces.) Can you make a map for the 3-sided pyramid (when its bottom face is removed)?

A more complicated pattern comes from the *dodecahedron*, which is made up of 12 identical 5-sided faces called *pentagons*. When you remove one face and flatten the rest out you get:



Dodecahedron

Most complicated of all is the map of the **soccer ball**! (I admit that I had some trouble doing this one myself!)



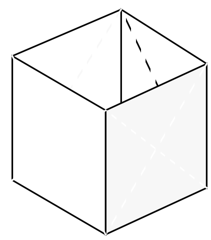
I leave to you the job of counting the number of edges and vertices in the above diagrams (and in your own diagram for the pyramid) and filling in the missing numbers in the table below. Notice that the last row is all 1's.—the number of faces plus the number of vertices is always 1 more than the number of edges!

	cube	pyramid	dodeca.	soccer ball
F=faces	5	3	11	31
E=edges	12			90
V=vertices	8	4		
F+V–E=	1	1	1	1

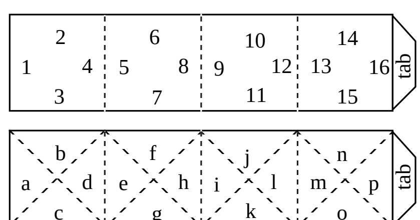
6. Turning a Cube Inside-Out

a paper-folding puzzle project

This is a little bit like a Rubik Magic puzzle, but is a *lot* cheaper! Fifty years ago a student at Princeton University by the name of Arthur Stone invented what he called the *flexatube*. It is just a cube of paper, missing the top and bottom squares. The puzzle is to turn the cube-tube inside-out by folding it only along the 4 vertical edges shown or along the 8 diagonal (dotted) lines connecting corner points.



You can make a flexatube by cutting a 2 and 1/2 inch wide strip of paper that is about 11 inches long. First mark one side with numbers and the other side with letters (as shown below) so you will be able to follow the hints in the next issue. Then, fold all the dotted edges back and forth several times, so they work easily. Use the tab to tape, staple, or paste the strip into the form of the flexatube, with the numbers on the outside.



other side

Answers and Solutions

for issue number 3

*1. Things with peculiar prices.*The boy was buying numerals at the hardware store. The number 157 needed three numerals 1,5,7 costing $3. The number 86 needed two numerals 8 and 6, costing $2. Each numeral costs $1. (Maybe they were putting up a new mail box, and their address was 157 86th Street.)

3. Cooking steak on the bar-b-que. Name the 3 steaks A, B, and C. Name their sides A1 and A2, B1 and B2, C1 and C2. There are 6 sides altogether, each needing 5 minutes to cook, for a total of 30 minutes. But 2 sides can be cooked at a time, so maybe half of 30, namely 15, minutes should be enough, if there were a way to have 2 steaks cooking on the bar-b-que for the whole 15 minutes.

Here is a method. First cook sides A1 and B1 for 5 minutes, then turn over steak A and replace B with C, cooking A2 and C1 for 5 minutes, then take A off (it is done), replacing with steak B, and cook B2 and C2 for 5 minutes, finishing both of them. This method takes only 15 minutes!

4. Boiling numbers down! The digital sum of 123456789 is 9. First, cross out the 9 (*cast it out*). Then, think, "1+2+3=6, 6+4=10, 10+5=15, 15+6=21, 21+7=28, 28+8=36, 3+6=9."

You can also boil down as you go along. That would go as follows: "1+2+3=6, 6+4=10, (1+0=1), 1+5=6, 6+6=12, (1+2=3), 3+7=10, (1+0=1), 1+8=9," where I have put the boil-down additions inside of parentheses ().

5. Moebius experiment. After one centerline cut, you still have a single strip, but it now has 4 half-twists in it. It has 2 sides, just like an untwisted loop. When you cut that strip, you get two loops that are linked together in a funny way. Each link still has 4 half-twists.

6. Seven link chain. Cut the third link in the chain. Then you have a single link (the one that was cut), a double link chain, and a 4 link chain. With a $1 piece, a $2 piece, and a $4 piece, you can make any amount up to $7. For example, 7=1+2+4, 6=4+2, 5=4+1, 3=2+1.

Here we have made any amount up to 7 by using only the numbers 1, 2, and 4, without using one more than once. Do you see how to make every number up to 15 by using only the numbers 1, 2, 4, and 8?

■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 5
FALL 1991

1. The Race to Mecca

a riddle

Old Mohammed died, and in his will he said that his two sons should race to Mecca on their camels. All his money would go to the son whose camel got to Mecca *last*. Of course, they didn't race very fast. Each wanted to get there last, so that he would win the fortune. So they delayed, dawdled, back-tracked, and basically made no progress towards Mecca. They were getting tired, hungry, thirsty, and generally worn out. Finally, they met a wise man who offered them some good advice. Then, they jumped onto the camels and rushed off at top speed.

What advice did the wise man give?

2. Cutting a Board

time out for some carpentry!

How many cuts are needed to cut an 8 foot long board into 8 pieces, each a foot long? How many cuts are needed if you are allowed to stack pieces and cut them all at once?

3. The Wolf—Goat—Lettuce Problem

finding the correct method

A man has a wolf, a goat, and a head of lettuce (don't ask why!). He must take them across the river, but he has only a small boat that will hold one animal or thing besides himself. He will have to take the wolf, the goat, and the lettuce across in separate trips. But he has a problem. If he leaves the wolf alone with the goat, the wolf will eat the goat. (That is not good.) If he leaves the goat alone with the lettuce, the goat will eat the lettuce. (That also is not good.) We assume that the wolf is not interested in eating the lettuce. By making trips back and forth, can the man eventually get all three across, without anything or anybody getting eaten?

Instead of drawing a picture of each stage of the operation, and instead of describing it in words, use the following notation, which "encodes" the situations that come up. Let W denote the wolf, let G denote the goat, let L denote the lettuce, let M denote the man, and let x denote water (we already used W for the wolf!). Then the starting situation is

MWGLxxxx

If the man takes the goat first, you will first write down the pattern WLxxMGxx (when he is still rowing across the river) and then WLxxxxMG when he arrives at the other side of the river. Your answer should be written down as a sequence of these 8 letter groups, and that is the way I will state the answer in the next issue.

4. Yet Another Secret Message

back to changing letters again!

There are 26 letters in the alphabet, so they will fit into 2 rows that are each 13 letters long. You can do this any way you want, and then those 2 rows can be your **secret key** for making and breaking secret messages. For my next secret message I will use the 2 rows below. For

```
ABCDEFGHIJKLM
ZYXWVUTSRQPON
```

example, whenever you see an A in the message, you substitute a Z; whenever you see a Q, substitute a J; whenever you see an M, substitute an N. Do you get it? Now see if you can decipher the following messages from William Shakespeare's **Macbeth**:

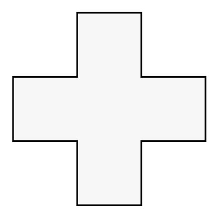
Wlfyov, wlfyov, glro zmw gilfyov,
uriv yfim zmw xzfowilm yfyyov.

Yb gsv kirxprmt lu nb gsfnyh, hlnvgsrmt drxpvw gsrh dzb xlnvh.

5. Turn a Cross into a Square

a problem in "geometrical dissection"
(in other words, cutting things up!)

A cross is made out of paper. It is 3 inches high and 3 inches wide, and has 1 inch thick arms.

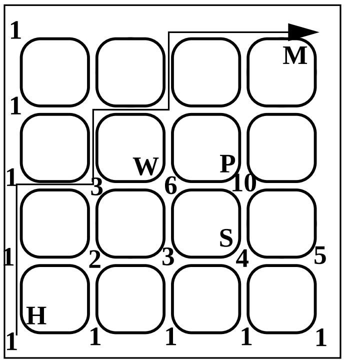


By making 4 straight cuts, cut the cross into 5 pieces that can be put back together to form a square.

6. Counting Paths

how many ways to reach a destination

A long time ago, when I lived in Salt Lake City, I used to walk to the Math Building from home every day. For variety I would take a different path each day. In the map below, my home (H) is on the lower-left corner and the Math Building (M) is near the upper-right corner. How many ways are there to go? (Assume that I never backtrack—namely, I go only north or east on each road.)

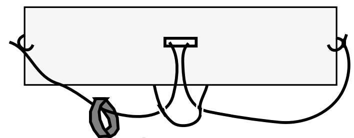


Hint: Sometimes in mathematics it is easier to solve a problem by doing *more* than you are asked for—as crazy as that sounds. In this case, instead of finding out only one number, the number of ways to go from H to M, you will find out the number of ways to get from H to each of the 25 road intersections, and you will write the answer in each intersection.

In order to go from H to some point P, you will take a path that goes through either W or S. That means that the number of ways of going from home to P is just the *sum* of the *number* of ways to get to W and the *number* of ways to get to S. In other words, if you already had written the number 6 at W and the number 4 at S, you could go ahead and put the sum, 10, at P. Start at home by putting a 1 there. (There is only one way to go from H to H without backtracking—stay put!) Put 1's at all the intersections exactly north of home, and all the intersections directly to the east of home, because there is only one way to get to each of these points. Use the W+S=P rule, slid around to other places, to fill in numbers at all the other points. For example, you first fill in the 2 as the sum of the number below it (1) and the number to the left of it (1). Then you fill in 3's above and to the right of the 2. You can go ahead and finish filling in all the numbers in the diagram. In the completed picture, the number you put on M (the Math Building) is the answer! (If you think *this* is complicated, then just try to draw pictures of all the different paths you could take. This would drive you crazy almost as fast as the problem earlier about finding 47 triangles!)

7. Make your own PUZZLE
a construction project

You will need a piece of cardboard, some string, and a ring or a washer. Cut a hole in the middle of the cardboard, making sure that the hole is too small for the ring to fit through. Punch small holes in both ends of the cardboard, so that you can tie the ends of the string. Put the puzzle together as shown below. The goal is to



move the ring from the left side string over to the right side string (from A to B). Good luck!

8. Pockets Full of Money
a silly game to play with money.

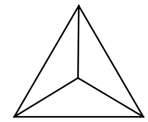
How many dollar bills would you need, in order to put a different amount of money in each of 5 pockets?

Hints, Answers, and Solutions
for issue number 4

1. Mrs. Trip's Triplets. 8. Once she has taken 2 hats of each of the 4 colors, she must get a triple hat match on the very next hat.

4. The worm and the encyclopedia. The worm travels only 2 inches. We assume that the volumes are arranged left to right, A through Z. The first page of volume A is right next to volume B, and the last page of volume D is right next to the right side of volume C, so the worm only has to go through two books, volumes B and C.

5. Maps of flattened polyhedra. The map of the pyramid (actually called a **tetrahedron**) is

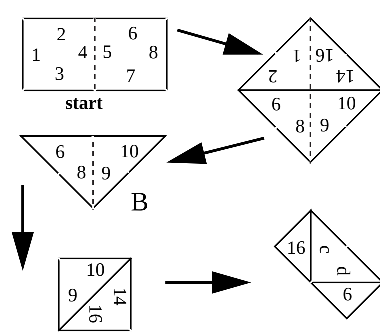


The completed table of face-edge-vertex counts is:

	cube	pyramid	dodeca.	soccer ball
F=faces	5	3	11	31
E=edges	12	6	30	90
V=vertices	8	4	20	60
F+V-E=	1	1	1	1

Try to map a basketball, and see what numbers you get.

6. Turning a cube inside-out. There are several ways to do this. I'll give you a few hints on how to do it one way. Start with the numbers on the outside. Fold two of the diagonally opposite corners on the top of the cube towards each other, at the same time that the other pair of corners on the bottom come together. This gives something flat that looks like the picture A below. Fold back along the horizontal line and get the "boat" in picture B. Look into the top of the boat and see 2 flaps. Push one to one side and the other to the opposite side, and push the 2 ends of the boat together, obtaining a square with pockets on both sides (picture C). Open both pockets and get picture D, which is really a smaller cube-tube. Fold this flat the other way, reverse the previous steps, and you will arrive at the original cube-tube, but it will now have letters on the outside.



HOPEWELL ELEMENTARY SCHOOL

MATH MINDS IN MOTION

NUMBER 6

FALL 1991

1. The Three Suspects

a problem in logical thinking

A burglary took place in which a valuable painting was stolen. The police suspected that the culprit was one of the following people: Alice, Bob, and Clyde. The police brought them all in for questioning. Alice said, “I didn’t do it.” Bob said, “Alice is lying.” Clyde said, “I didn’t do it.”

I tell you the following two facts:

1. One of the three suspects is the burglar.

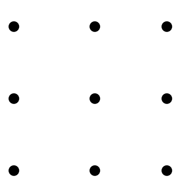
2. Exactly one of them is telling the truth.

Now tell me!—who is the burglar, and who is telling the truth?

2. Dots and Lines

a problem in geometrical perception

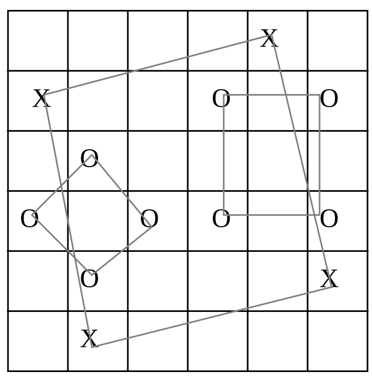
Draw 4 straight lines, without lifting your pencil from the paper, and without retracing any of your steps, so that you go through all 9 points below:



3. The Game of HIP

*a “hip” person does **not** want to be “square”!*

HIP is a game that is a little like Tic-Tac-Toe. You start with a 6 by 6 grid. One person starts by marking an X in one of the boxes, and the other player follows by marking an O in another box. From then on they take turns until someone forms the 4 corners of a square with their letters. Whoever forms a square first LOSES (but only if the other player announces victory and shows where the square is). The squares can be tilted. Below are 3 examples:



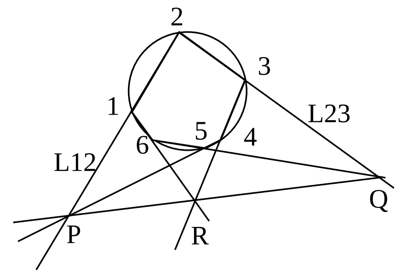
Can you think of a simple strategy for the O player to follow, so that she cannot lose?

4. Pascal’s Theorem

an important fact about circles

When the famous French thinker Pascal was 16 years old (in the year 1639) he discovered the following fact about circles and lines.

Start with a circle. Mark any 6 points on



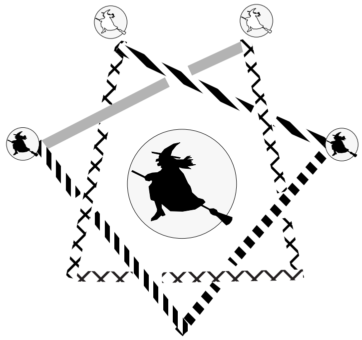
the circle and label them 1, 2, 3, 4, 5, 6. Connect points 1 and 2 by a line called L12, points 2 and 3 by a line called L23, and so on for lines called L34, L45, L56 and L61, forming a **hexagon**. Extend the lines L12 and L45, which are opposite sides of the hexagon, until they meet at a point P. Similarly, find the intersection of the lines L23 and L56, a point Q. Finally, let R be the point of intersection of the last pair of opposite sides, L34 and L61**.** Pascal discovered that the straight line connecting P and Q always passes through the point R as well.

This is one of the basic results in the mathematical subject called **projective geometry**. It is also true if you replace the circle with any projected image of a circle. These give curves called **ellipses**, **hyperbolas**, and **parabolas**. Ellipses are the exact shapes of orbits of planets about the sun, and parabolas are used in shaping telescope mirrors for looking at stars.

5. Four Witches on a Heptagon

a Halloween rearrangement

Two white-robed witches sit in their homes at the two northern-most *fly-way* turns. (A *fly-way* is a freeway for witches only!) Two black-robed witches also sit at home, as shown below.

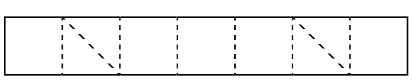


On Halloween night, the white witches decide to go to the black witches' homes to play tricks on them, and the black witches decide to fly to the white witches' homes to play tricks on them. They all must fly on the fly-way. One *move* consists of flying one of the witches along one of the straight sides of the 7-gon (*heptagon*) to another corner. They all must arrive at their destinations without ever running into each other on the fly-way. How many moves in all does it take for them to switch places like this?

6. Paper Cube Puzzle

cutting and folding paper again!

Measure and cut a strip of paper that is 1" wide and 7" long. Mark the inches along the side and fold at each mark, making a crease. Also crease the 2 diagonals shown below.



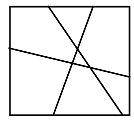
Using only the folds shown above, make the paper strip into a paper cube!

7. Cutting Paper into Lots of Pieces

a problem in finding patterns

Sometimes general facts are discovered by looking at particular examples and seeing a general pattern.

If you draw 1 straight line across a rectangular piece of paper, it divides the paper into 2 regions. If you draw 2 crossing lines you make 4 regions. But, if you draw 3 lines, you will get at most 7 regions (not 8). Do you see any pattern in



the sequence 1, 2, 4, 7? If so, you can predict the largest number of regions you can get by using 4 lines. Try it and see if your guess was right! Also, how many pieces can you get by using 5 lines? If you are feeling ambitious, see if you can predict how many regions you would get with 10 cuts. I'll report next time on how many pieces you would get if you did 100 cuts.

Answers and Solutions

for issue number 5

1. The Race to Mecca. The wise man suggested that each man should ride on his brother's camel instead of his own. Then each man races to get to Mecca first, so that his own camel will get there last!

2. Cutting a board. Seven cuts will suffice to make 8 pieces. If you can stack pieces between cuts, then you can make do with only 3 cuts: first you cut the board in half, then you stack and cut in half again, finally you stack the 4 pieces and cut in half again.

3. The Wolf-Goat-Lettuce Problem. Remember that M is the man, xxxx is the water in the river, G is the goat, W is the wolf, and L is the head of lettuce.

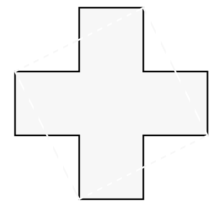
```
MWGLxxxx,
WLxxMGxx,
WLxxxxMG,
WLxxMxxG,
MWLxxxxG,
LxxMWxxG,
LxxxxMWG,
LxxMGxxW,
MGLxxxxW,
GxxMLxxW,
GxxxxMWL,
GxxMxxWL,
MGxxxxWL,
xxMGxxWL,
xxxxMWGL.
```

4. Yet another secret message. These are two things that the witches in **Macbeth** say.

Double, double, toil and trouble,
Fire burn and cauldron bubble.

By the pricking of my thumbs,
Something wicked this way comes.

5. Turn a Cross into a Square. Cut along the 4 lines shown below and form the square with the dotted-line border.



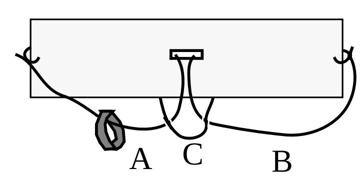
6. Counting paths. 70. The numbers appearing in this problem are called **binomial coefficients**. They are most frequently written in the famous (to mathematicians!) **Pascal triangle**:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Only the first 7 rows of the Pascal triangle appear above. The "real" triangle goes on forever! Notice that each number is the sum of the two closest numbers above it.

7. Make your own PUZZLE. This is a very tricky puzzle. Even though I have solved it, it still looks impossible to me! Your parents should also think it cannot be done—surprise them!

Move the ring along string A until it is close to the hole in the cardboard. Then pull the string in front of the hole towards you until the loop C comes around behind and out through the hole. Then move ring more to the right. Pull the C loop back into the hole and back where it started. Now the ring should be on string B.



8. Pockets Full of Money. You might think that you need $15, in order to put 1,2,3,4, and 5 bills into the 5 pockets. However, you need only $10, because you can put 0 bills in the fifth pocket instead of 5. (See—not *all* the pockets need to be full of money!)

■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 7
FALL 1991

1. The Two Coins
an exercise in exact English

Two coins add up to 30 cents, yet one of them is not a nickel. They are both standard, common, modern American coins. What are the two coins?

2. Two Jugs of Water
a measuring problem

You have only a 3 quart jug and a 5 quart jug, with no measurements marked on them, and you are supposed to use them to measure out exactly one quart of water for your science experiment. How do you go about this?

You are allowed to completely fill either one at the faucet (any number of times) and you are allowed to pour one into the other, but you must keep pouring until one of the jugs is empty or full.

3. The Cup and the Rope
a topological magic trick

Take a piece of rope or thick string at least a meter long, fold it in two at its middle, and tie it to the handle of a mug as follows: insert the loop through the handle, then take the other end of the doubled rope and thread it through the loop, forming a knot. Hand the ends of the rope to a member of your audience (a friend, sibling, or parent), cover the mug with a towel, and say "I will now use magic to untie the knot and release the mug!" Reach under the towel, untie the knot, and bring out the mug. Figure out how you are going to do this!

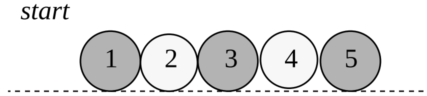
4. The Monk Goes up the Mountain
and meets himself coming down?

A monk spends the first day hiking up to the top of a mountain along a trail. He sleeps overnight at the top. The next day he hikes back down the trail to the bottom. On the second day will he reach some spot which he reached at the same time the previous day?

5. Sliding Pennies and Dimes
a combinatorial puzzle

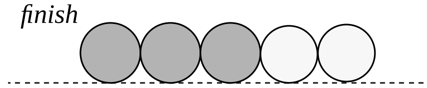
Get 3 pennies and 2 dimes and lay them side-by-side in a line.

start



Start off with them alternating penny-dime-penny-dime-penny (which we write PDPDP). Slide two coins at a time and in three moves get the order PPPDD (and all coins adjacent). Each pair of

finish

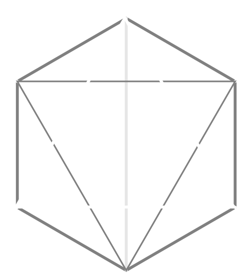


coins that you slide must be adjacent and you must keep them in the same order with respect to each other. For example, you are allowed to put two of your fingers on top of coins 3 and 4, slide them upwards and then over to the left and back down to the left of coin 1. You are **not** allowed to put your fingers on the dimes (2 and 4) push them upwards, bring them together, and put them at the right end, next to 5. (This is not allowed because 2 and 4 are not adjacent to begin with.) You *are* allowed to just shove coins 1 and 2 to the left, leaving a gap in which to fit 2 other coins later. Hint: the group of coins is allowed to end up at a different position along the baseline.

6. The Club of Six
an example of Ramsey Theory

Six kids form a club. Unfortunately, any pair of kids either hate each other or love each other. If there is no group of 3 kids that all hate each other, then there must be some group of 3 that all love each other. Meanwhile, it works the other way around: if there is no group of 3 that all love each other, then there must be a group of 3 that all hate each other.

Another way to think about this fact is to apply it to coloring the lines in a diagram of a hexagon. Put in lines connecting



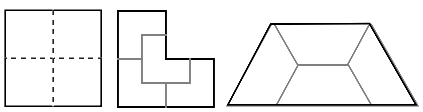
each pair of the 6 outside points. There are 15 lines in all. If you color each line either red or green, then you are guaranteed to find 3 of them forming a red triangle or 3 forming a green triangle (or both!).

■
PRINCETON AVENUE
HOPEWELL, NJ 08525

7. Cutting into Equal Parts

another geometrical dissection problem

I learned this problem from the book *Aha!* by Martin Gardner. First I show you several examples of how to cut up shapes into 4 pieces that are all alike. For example, see below that the square divides into 4 squares, the L shape divides into 4 smaller L-shaped pieces, the trapezoid divides into 4 trapezoids. Now see



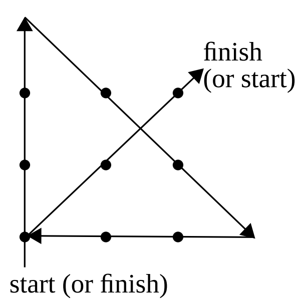
if you can divide a square into 5 pieces that all have the same shape as each other, and are all the same size, but are not squares.

Answers and Solutions

for issue number 6

1. The Three Suspects. Either Alice is telling the truth or she is lying. If she is lying, then Bob is telling the truth. If she is telling the truth then Bob is lying. Either way, one of them is telling the truth. Because we know that only one of the three suspects is telling the truth, that means that Clyde is lying. Hence, he is the burglar and Alice is telling the truth, because she is not the burglar.

2. Dots and Lines. The tricky thing is that the turns are not made at the dots!

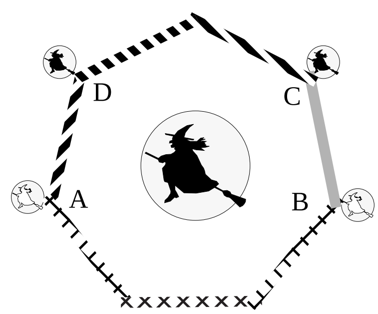


3. The Game of HIP. The strategy guaranteed to keep you from losing is also guaranteed to annoy your opponent! You just copy what he does, only on the opposite side of the board relative to the center point of the board. Below is a sample of the start of a game. Notice that O's

X2		O4		O3	
		X1			
			O1		
	X3		X4		O2

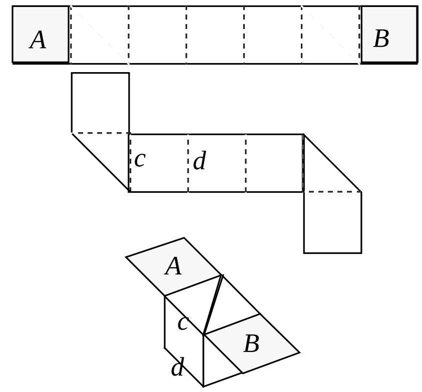
first move, at O1, is opposite, with respect to the center, of X's first move, at X1. Similarly, O's second move, O2, is opposite X2. And so on. Since your position stays the same as his, you cannot form a square and lose, because he would have formed the "opposite" version of the same square first.

5. Four Witches on a Heptagon. The confusing thing about this puzzle is the twisted shape of the fly-way. The first step in solving something like this is to unwind the picture so it is easier to think about. Here is what you get



Now it is easy to see that the witches must fly around the same direction, say clockwise, with A going to C, B going to D, C going to A, and D going to B. Thus, they must use 14 moves.

6. Paper Cube Puzzle. First fold along the diagonal creases. Then fold up along each of the middle 4 creases, each one by an angle of 90 degrees, bringing the flaps A and B up to the top, with the triangles matching to form the square on the top of the cube. Note where the lines *c* and *d* end up. Finally, fold each of the flaps A and B down, completing the cube.



7. Cutting Paper into Lots of Pieces. The pattern in the sequence 1, 2, 4, 7 is that you add 1 to get from 1 to 2, you add 2 to get from 2 to 4, you add 3 to get from 4 to 7. To get the next number you would add 4, obtaining 11. This is the number of regions you can get with 4 cuts. The next number in the sequence is 11+5=16, the number of regions you get with 5 cuts. For 10 cuts, you would add (to 16) the numbers 6, 7, 8, 9, and 10 getting 56.

A general formula for the greatest number of regions you can get with N cuts is N x (N+1) / 2 +1. For 10 cuts you get 10 x 11 = 110, 110/2 = 55, 55+1 = 56, the answer we obtained in the previous paragraph. For 100 cuts you get 100 x 101 = 10100, 10100/2 = 5050, 5050+1 = 5051. (This is also one more than the sum of all the counting numbers less than or equal to 100.)

8. The Trohee and the Falmo

a classic brain teaser

A remote island has two tribes. The Trohee always tell the truth, and the Falmo always lie. You meet 3 tribes-people, whose names are Albacore, Banana, and Coriander. You ask Albacore if she is a Trohee or a Falmo. You cannot understand her mumbled reply. You then ask Banana what she said. Banana says, "Albacore said she is a Falmo." Then Coriander said, "Don't listen to Banana—he is not telling the truth." From what tribes do Banana and Coriander come?

HOPEWELL ELEMENTARY SCHOOL

MATH MINDS IN MOTION

NUMBER 8
FALL 1991

1. The Man in the Painting
the first brain teaser I remember hearing

(There are many variations on this type of riddle, but this is probably the most common one.)

A man is looking at a portrait on the wall. He makes the following statement about the person in the picture:

> "Brothers and sisters have I none, but that man's father is my father's son."

The man is looking at a picture of whom? (A discussion of the solution appears later in this newsletter. As is often the case here, you may find the solution hard to understand even after you have solved the problem for yourself!)

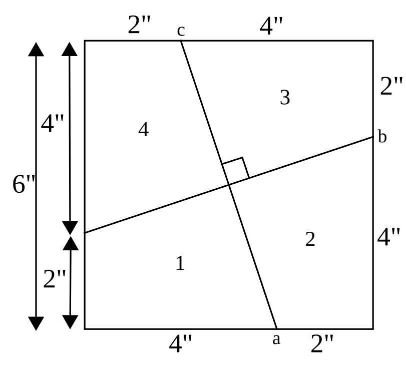
2. Three Jugs of Water
a harder problem in measuring liquids

You have 8 quarts of apple cider in a 2 gallon jug. (Recall that one gallon = 4 qts.) You wish to divide the cider equally with a friend who has come to you with a 5 qt. jug. (She is making things difficult—why didn't she bring a 4 qt. jug?) You also have a 3 qt. bowl that is clean, but all your other measuring bowls and cups are dirty. Your job is to pour cider back and forth among the 8, 5, and 3 quart containers until the 8 and 5 quart jugs both have exactly 4 quarts of cider. As before, you must keep pouring until either the from-container is empty or the to-container is full.

You may find it helpful to write down your solution as a sequence of 3-digit numbers, each 3-digit number encoding the number of quarts of cider currently in the 8, 5, and 3 quart containers. For example, the problem starts in the **position** 800 (8 quarts of cider in the 8 quart jug, and none in the other two). You want to end up with the position 440 (cider split equally between the 8 and 5 quart jugs). Say you started off by filling the 5-jug from the 8-jug, filling the 3-bowl from the 5-jug, and then emptying the 3-bowl back into the 8-jug. You would write that down as the sequence 800, 350, 323, 620. The answer I know uses 7 pours.

3. Making Two Squares from One
a Pythagorean puzzle

Draw a straight line through the center of a square piece of paper, and then draw another line *perpendicular* to the first line (and also through the center). (Perpendicular means that the two lines meet at a "square" angle, what geometers call a *right angle*.) For example, if your square is 6" by 6" your lines could divide the borders into 2" and 4" lengths.

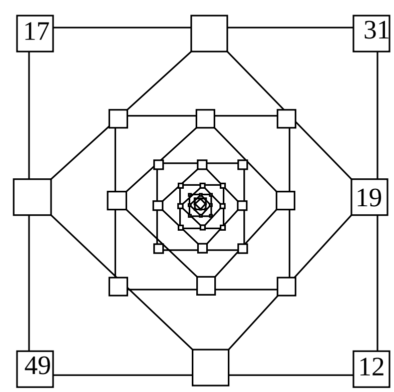


Cut along the lines, dividing the square into 4 pieces all alike. Now see if you can rearrange those four pieces to make 2 squares. This is really a "trick" problem—one of the squares will be just a square-shaped empty space inside of the other square.

Another hint if you want it: There is a solution in which the pieces 1 and 2 still touch each other at their corner a, pieces 2 and 3 still touch at their corner b, and pieces 3 and 4 still touch at their corner c.

4. Diffy Convergence
a math experiment in repeated subtraction

I have been informed recently that a **diffy** is a kind of diagram in which subtractions must be used to fill in the blanks. Here is an example:



Your job is to fill in each blank with the difference of the two numbers that the blank lies between. For example, I have already filled in the blank 19 with the difference of 31 and 12. Eventually you will get a square of zeros and then you can stop. Experiment with different starting numbers and see if you always end up with a square of zeros. For bigger numbers you might want to use a calculator. I was surprised to discover that, when I picked bigger starting numbers, it didn't seem to increase how long it took before I reached a stopping point (where the diffy *converged* to all zeros). For a math science fair project, you might keep a record of all your experiments, and see if you can find some sets of small numbers that take an especially long time to converge. (I found 4 numbers 0, 2, 6, 13 that took 10 "squares" before reaching all zeros.)

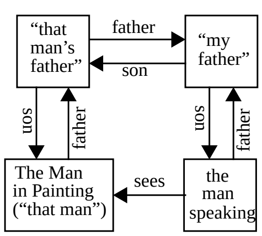
Discussion of "The Man in the Painting"
answer to problem 1

The usual solution starts by figuring out to whom "my father's son" refers. Since the man has no brothers and sisters, "my father's son" must be the man himself. Then we can rewrite his statement as follows:

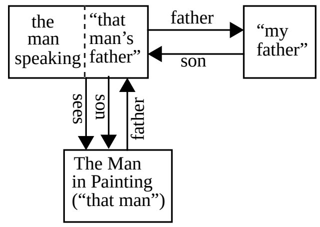
"That man's father is me."

Namely, the man in the painting is his son.

The usual solution is fine for people with strong verbal skills, but I prefer a method of solution involving the *diagramming* of the problem. First we make boxes for each person mentioned, then draw arrows for all the relations we know, like "son of", "father of", "sees painting of", and so on. To get from a man to his fa-



ther, follow the arrow labeled "father", and to get from a man to his son, follow an arrow labeled "son". Now you can see that the two boxes labeled "that man's father" and "the man speaking" are both sons of "my father", so they must be brothers or the same man. But the man says he has no brother. Hence, "that man's father" and "the man speaking" are the same, and the two boxes can be combined, as we have done in the new diagram appearing below. From this dia-



gram you can now observe that "the man in the painting" is a son of "the man speaking."

Comment. Although the diagram method looks complicated, and for this problem it is probably *overkill*, it really helps for harder problems where it becomes very difficult to hold all the given facts in your head. Once you have diagrammed all the facts, you can often "let the diagram do your thinking for you!" When you study **algebra** something similar will happen. Each fact will translate into an "algebraic equation", and you can then manipulate those equations until they tell you the answer!

Answers and Solutions
for issue number 7

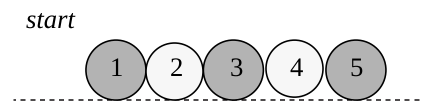
1. The Two Coins. The two coins are a quarter and a nickel, adding up to 30 cents. The *quarter* is the one that is not a nickel. (Some people think this trick is unfair, but I don't!)

2. Two Jugs of Water. Fill the 3 quart jug, then empty it into the 5 quart jug. Now there are only 2 quarts capacity left in the 5-er. Fill the 3-er again, and pour it into the 5-er until it is full. There will be exactly 1 quart left in the 3-er.

3. The Cup and the Rope. The "knot" really is not a knot at all. If you loosen it up, you can form a loop big enough to go around the body of the cup, and then out the hole in the cup handle.

4. The Monk Goes up the Mountain (and meets himself coming down?) Yes—on the second day, coming down the mountain, the monk must reach some spot that he reached at the same time the previous day. Here is an easy argument to convince you. Suppose a second monk starts up the mountain on the second day, copying exactly the first monk's progress from the first day, taking the same time to get to every spot, starting and finishing at the same time. Then you should agree that these two monks, one going up and one going down, have to meet each other somewhere along the way. That place is the place where the first monk was at the same time on the first day!

5. Sliding Pennies and Dimes. I know 2 solutions to this one.

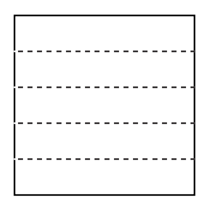


Solution 1. Step1— Slide coins 1 and 2 left, leaving a space for 2 coins. Step 2—Fill that gap with coins 4 and 5. (Now you have the order 12453, which is PDDPP.) Step 3— Move the 2 pennies from the right end to the left end. Done!

Solution 2. Step 1—Slide coins 3 and 4 around to the left of coins 1 and 2, leaving a 2 coin gap between coins 2 and 5. Step 2—Fill that gap with coins 4 and 1, leaving a 2 coin gap between coins 3 and 2. (Now you have 3-gap-2-4-1-5, which is P-gap-D-D-P-P.) Step 3—Fill the gap with the 2 pennies on the right end. Done!

A harder version of this puzzle demands that every pair of coins you move be a dime-penny or penny-dime. This version requires 4 moves.

7. Cutting into Equal Parts. The square can be divided into 5 rectangles that are all alike.



8. The Trohee and the Falmo. First, you can figure out for yourself what Albacore mumbled. If she were a Trohee, she would tell the truth and say she is a Trohee. Meanwhile, if she were a Falmo, she would lie and also say she is a Trohee. *In either case, she would not say that she is a Falmo.* Second, let us consider what Banana said: "Albacore said she is a Falmo." Hence, Banana is lying, so he is a Falmo. So, Coriander is telling the truth about him, and Coriander has to be a Trohee. We cannot figure out what Albacore is, outside of being a tuna fish.

■ HOPEWELL ELEMENTARY SCHOOL ■

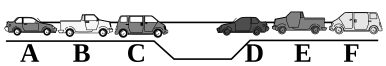
MATH MINDS IN MOTION

NUMBER 9
FALL 1991

1. The Very Narrow Road

another combinatorial puzzle

Six cars are traveling along a very narrow one-lane road. On one side is a rock cliff-face and on the other side there is a sharp drop-off. Three cars are going east, and three are going west. They all meet at a spot where it is too narrow to pass, and they come to a stop. There is a small turnout between the cars, but the turnout is only big enough to hold one car. How can the cars get by each other?



As usual, we devise a notation (or *code*) for each position we encounter. Let's use a small letter for the car in the turnout. From the starting position ABCDEF, suppose that car C goes into the turnout, giving the position ABcDEF. Now D can pass C, giving us ABDcEF. If A and B back up, then both E and F can pass C as well, giving us ABDEFc, and then c can go on his way, leaving ABDEF. The rest is up to you!

2. Jacqueline and the Nuns

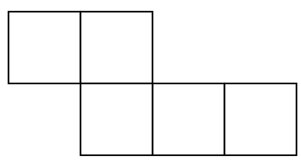
trying to line the girls up

Many of you have heard of Madeline and the orphans who parade around Paris in "two straight lines." In another orphanage for girls live Jacqueline and some number of other girls, but certainly fewer than 84! One day the nuns line them up in groups of 3 to form 3 straight lines. Unfortunately, Jacqueline is left over and must walk by herself at the end. The next day, the girls are lined up by fours, to form 4 straight lines. Again, poor Jacqueline walks alone at the end. On the third day, the nuns try lining the girls up in groups of 5, and it works perfectly. How many girls are there?

3. Another Toothpick Puzzle

in case you thought we were finished with these guys forever!

In the diagram below are 16 toothpicks arranged to make 5 small squares:



How can you move exactly 2 toothpicks to other spots in the picture and produce a diagram containing exactly 3 small squares and one larger square? (No doubling up of toothpicks allowed!)

4. Counting Handshakes

the more the merrier!

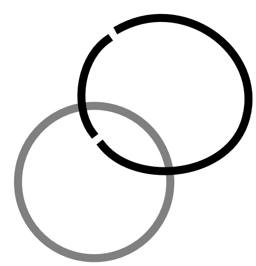
If 2 people shake hands, there is 1 "handshake" between them. With 3 people there are 3 handshakes. With 4 people there are 6 handshakes. How many handshakes do 7 people make among themselves?

Notice that when you add one more person, that person must shake hands with all the people so far. So, if there are 4 people who have already shaken hands (6 shakes) and one new person enters the scene, then that person shakes 4 hands, for a total of 10 shakes for 5 people. Now all you have to do is find the number for 6 and then for 7 people.

5. The Borromean Rings

three linked rings that come apart when one is removed

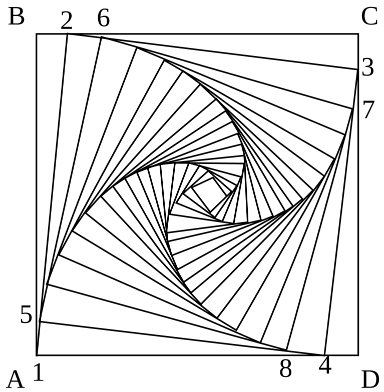
You've probably heard of musical groups that fall apart when one of the band members leaves. Something similar can happen in topology. Three rings can be put together in such a way that none can be removed from the others. However, if you cut one ring and take it away, then the other two rings will not be "linked" and can be separated. Here is a simple example, called the **Borromean Rings**:



5. Spiraling Squares Doodle

and the four-corner bug chase

In one of my favorite doodles I start by drawing a square, and draw lines in a clockwise fashion connecting the current point with points slightly further in along the side after the next corner. In the following diagram, I have numbered the points in the order I come to them.



If you could keep this up forever, the four spirals would each go around the middle point infinitely many times.

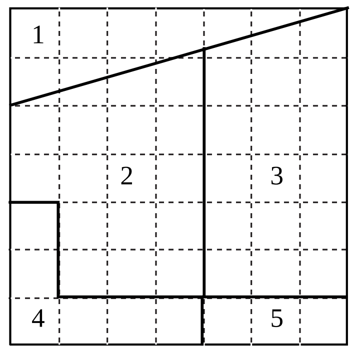
Suppose that 4 ladybugs A, B, C, D start at the four outside corners and proceed to chase each other (slowly, all with the same speed) so that A is always heading directly towards B, but B is chasing towards C, and C towards D, and finally D is chasing A. Then each will end up following a spiral path, just as we have drawn above. If the outside square is 2 inches on a side, and the ladybugs crawl at 1 inch per 10 seconds, how long does it take for them to meet at the middle of the square?

Once you have determined how long the chase takes, you can determine the length of each spiral.

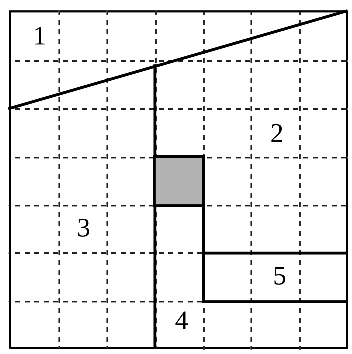
6. The Disappearing Square

cut a paper square into 5 parts that can be rearranged to make the "same square" but with a hole in the middle

Martin Gardner's *Entertaining Mathematical Puzzles*, a book reprinted by Dover in 1986, contains the following paradox. You shouldn't be able to change the total area of a sheet of paper, just by rearranging its pieces—but, in the example below, it seems that the area decreases from 49 square inches to 48. Construct a 7" by 7" square of paper, and cut it into 5 pieces as shown below.



Then you can rearrange the five pieces so that the middle square disappears:



How do you explain this?

7. More on Diffy Convergence

a mathematical research report

In the preceding issue of this newsletter we talked about the **diffy**, where you start with 4 numbers, say 4 2 5 15, then take a "diffy" step by writing down the differences between the first and second numbers, the second and third numbers, the third and fourth numbers, and the fourth and first numbers. After one diffy step, from 4 2 5 15, you would get 2 3 10 11. Continuing with a series of diffy steps, you would get 1 7 1 9, then 6 6 8 8, then 0 2 0 2, then 2 2 2 2, then 0 0 0 0. In this case it took only 6 diffy steps to reach all 0s.

My friend David Robbins and I did some **mathematical research** (over lunch) on the question of why most starting numbers converge to zero rapidly, and how to make starting numbers that take a long time to converge. I report two of these results here, without proof.

Result 1. If your starting numbers are *a*, *b*, *c*, *d*, and if both *a* and *c* are smaller than both *b* and *d*, then the diffy converges to zero in 4 steps.

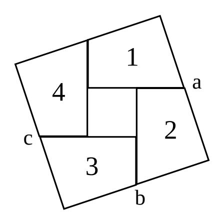
Result 2. *How to make a long-lived diffy.* Start with the 3 numbers 0 0 1 and make a sequence of numbers by forming each new number as the sum of the latest 3 numbers in the sequence. You get the sequence 0 0 1 1 2 4 7 13 24 44 81 149... If you go out N steps in this sequence and take 4 consecutive numbers to be the start of your diffy, then it will take approximately N plus half N diffy steps before the diffy converges to zero. For example, 2 4 7 13 takes 9 steps to converge, while 7 13 24 44 takes 12 steps, and 24 44 81 149 takes 15 steps to converge. Consequently, if you want to drive you parents crazy, challenge them to do a diffy with the start 0 35890 101902 223317— it should take 34 diffy steps! That should keep them busy!

Answers and Solutions

for issue number 8

2. Three Jugs of Water. Here is the answer, written in the 3 digit code: 800, 350, 323, 620, 602, 152, 143, 440. In English: 1) fill the 5-jug with cider from the 8-jug; 2) fill the 3-qt bowl from the 5-jug; 3) empty the bowl back into the 8-jug; 4) pour the 2 quarts of cider in the 5-jug to the bowl; 5) fill the 5-jug from the 8-jug; 6) fill the bowl by pouring one quart from the 5-jug, leaving 4 quarts there; 7) dump the bowl back into the 8-jug.

3. Making Two Squares from One. You should obtain the following picture. Notice that the inside and outside



squares are not lined up at the same angle.

4. Diffy Convergence. Instead of drawing the picture, I will just write down the 4 numbers in a square in clockwise order. Starting with the diffy 17, 31, 12, 49, we obtain 14, 19, 37, 32, then the square 18, 5, 18, 5, then 13, 13, 13, 13, then 0, 0, 0, 0. It takes only 4 "steps" to converge to zero.

■ HOPEWELL ELEMENTARY SCHOOL ■

MATH MINDS IN MOTION

NUMBER 10
FALL 1991

Editor's Farewell

and recommendations

I hope you enjoyed these problems as much as I did. If you have discovered that you like mathematical puzzles and games, you might enjoy reading one of the many books that are devoted to such entertainments. Bookstores usually have such books in a "Games" section which will also have books on board games, card games, crossword puzzles, and so on. In particular, I can recommend two such books that I have found recently in one of our local stores. The first is Martin Gardner's *Perplexing Puzzles and Tantalizing Teasers* (Dover $3.95). The second is more mathematical (closer to the level of this newsletter): Martin Gardner's *Entertaining Mathematical Puzzles* (Dover $2.95). Gardner has many excellent books. If you happen to buy one that is too difficult, you can always grow into it! (The same can be said for this newsletter. If you found some of the problems and solutions too difficult, look at them a few years from now—maybe they will not seem as hard. (I hope!)

1. The Non-Transitive Dice

sometimes there is no "best" team

People often talk about how team A is "better than" team B. What they mean usually is that if A and B play a lot of games, then they expect A to win more often than B does. What people may not realize is that there may not really be any "best" team among 3 or more teams. For example, team A can be better than team B, and team B can be better than team C, but team C can be better than team A! Pretty shocking, eh!

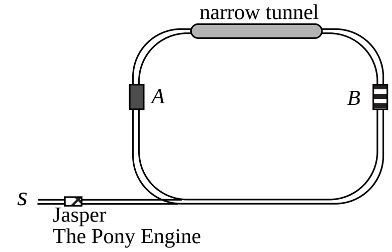
To demonstrate how this can happen, I will show you 4 dice (each is a cube with numbers on each of its six sides) invented by Bradley Efron, a professor at Stanford University. Think of each of the dice as being a "team". A game will consist of rolling 2 dice and seeing which one has the larger number on top. We shall call the dice A, B, C, and D. It will turn out that A will beat B, on average, 2 out of 3 games. Similarly, B will beat C, C will beat D, and D will—surprise!— beat A. So, if you wanted to have fun with someone, you could show him the 4 dice and let him choose whichever one he wants to roll with. Then, going second, you can pick the one that will win 2/3 of the time! For example, if he chooses A, then you pick D, and if he chooses D, you pick C. If you keep track of how many times each of you wins, you will see that in the long run you win twice as often as he does!

Here are the numbers to put on the Efron dice: A) 0,0,4,4,4,4; B) 3,3,3,3,3,3; C) 2,2,2,2,6,6; D) 1,1,1,5,5,5.

2. The Pony Engine's Puzzle

a railway switching problem

The little pony engine Jasper is given the job of interchanging the positions of the two big railroad cars, Abigail (*A*) and Bruce (*B*), and then returning to his home in the siding *s*. Neither of the big cars can fit through the narrow tunnel, but Jasper himself can go through. Jasper can hook up cars at either end and he can push or pull two cars at the same time. Can you figure out how he can accomplish his job?



3. A Mathematical Card Trick

guessing the color of playing cards

If you have ever studied a deck of playing cards, you probably know that there are 52 cards (not counting the 2 jokers), 13 of each *suit*. The suits are spades, hearts, diamonds, and clubs. The spades and clubs are *black* and the hearts and diamonds are *red*.

The effect. You present a deck of cards to a member of the audience (the *victim*), who is instructed to shuffle the deck once and put the deck facedown on the table. You and the victim each take a card from the top of the deck, not showing the other what his card is. You tell the victim the color of his card. Then you both take another card off the deck, and again you guess the color of his card. And so on.

How to do the trick. Ahead of time you have prepared the deck so the cards alternate red and black in color. (Don't call this to the attention of your audience!) Put the deck facedown in front of your victim and tell him to "cut" the deck into 2 parts of approximately the same size, forming 2 facedown piles. Turn over the top 2 cards, saying some nonsense like:

"Let's see if you did a good job." If the cards are different colors, say "Good!" and set the cards aside. If the cards are the same color, put one back on its pile, set the other aside, and say "That's better!" (The whole purpose of this rigmarole is to make sure that the two piles start with different colors, which is what makes the trick work.) Tell your victim to shuffle the 2 piles together, using a *riffle* shuffle. (If neither you nor your victim knows how to do a riffle shuffle, just leave the two piles alone, and whenever one of you needs to take a card during the trick, take it off the top of *either* pile!) Instruct your victim to take a card off the top of the deck and keep it hidden. You take a card, look at it, and put it facedown on the table. (Never show anyone your cards!) If your card is red, tell your victim that his card is black, "because you can read his mind." If your card is black, tell him his card is red. Have him put the card face up on the table so the audience can see that you are correct. You both continue drawing cards (and you guessing his card) until you are satisfied that the victim has suffered enough!

The Explanation. One pile of cards, from the top down, has the order *RBRBRBRBR....*, where *R* is a red card and *B* is a black card. The other pile is alternating red-black also, but starts with black; that pile we'll write as *brbrbrbr...*If you *don't* shuffle the two piles together it will be easier to see how the trick works. (All the single riffle shuffle does is choose in advance which piles you and your victim will take cards from.) Notice that if you both take cards from the *RBRB...* pile, you will have opposite colors. If you both take cards from the other pile, again you will have opposite colors. Finally, if you take cards from different piles, you will have opposite colors. So, you can always guess the color of his card, because it is the opposite color to yours. Furthermore, after you each pick one card, the two piles remaining are ready for the same trick to work again. Every pair of cards taken will be of opposite colors. (This is called the **Gilbreath Principle**.)

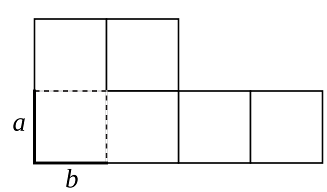
Answers and Solutions
for issues 9 and 10

1. The Very Narrow Road. The cars that remain go back and forth in front of the turnout, with the car in the turnout being released after each move of the shuttle. In the notation we devised, one solution goes as follows. Recall that the cars ABC are trying to drive to the right and that the cars DEF are trying to go left. From the starting position ABCDEF, suppose that car C goes into the turnout, giving the position ABcDEF. Now A and B back up, DEF go forward, giving us ABDEFc, and c goes on his way, leaving ABDEF. Then F backs into the turnout, DE back up, AB go forward, until the position fABDE, so F can go on his way. Then A backs into the turnout, BDE pass him, giving position BDEa, and A goes on his way. The remaining positions are BDE, BDe, eBD, BD, bD, Db, b, all gone!

2. Jacqueline and the Nuns. There are 25 girls. When you divide by 3 you have a remainder of 1 (Jacqueline), when you divide by 4 you also have a remainder of 1 (poor Jacqueline), but 25 is evenly divisible by 5. Also, 25 plus any multiple of 60 would also work (because 3, 4, and 5 all divide 60 evenly), but 25+60 equals 85 and the problem stated that there were fewer than 85 girls.

Digression. A **systematic** way to solve such problems without thinking too hard or trying too many possibilities, is to play with the *arithmetic progressions* (or sequences of numbers) that are mentioned. For example, the first fact says that the number of girls is 1 more than a multiple of 3. That means that the answer is in the sequence 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31,... obtained by starting with 1 and repeatedly adding the number 3. Any sequence that you get by starting with some number, then repeatedly adding another fixed number, is called an *arithmetic progression*, and will be completely determined from the first two numbers in the sequence. For example, if I gave you the sequence 2, 9, 16, 23 and asked you for the next term in the sequence, you would notice that each term is obtained by adding 7 to the previous term.. Hence 23+7=30 would be the next term in the arithmetic progression. Now, consider the second condition to be satisfied: that there is a remainder of 1 when dividing by 4. Pick out the numbers from the 1, 4, 7, 10, 13, 16,... progression that satisfy the second condition. These are 1, 13, 25,... (every fourth number from the sequence), another arithmetic sequence, this time obtained by repeatedly adding 12. Run the sequence out a little further: 1, 13, 25, 37, 49, 61, 73, 85, 97,... When these numbers are divided by 5, you get the sequence of remainders 1, 3, 0, 2, 4, 1, 3, 0, 2,.... That is also an arithmetic sequence, except that you add 2 each time and subtract 5 if the number gets big enough. Since we want the answer to have remainder 0, we see that 25, 85 are possible answers. But any number in the arithmetic progression 25, 85, 145, 205,... starting with 25 and 85 would also be a possibility.

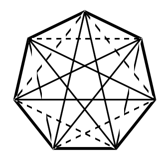
3. Another Toothpick Puzzle. Two toothpicks that were in the dotted locations are moved to their new positions *a* and *b*.



4. Counting Handshakes. We already determined that there are 10 handshakes among 5 people. When a sixth person arrives, he has to shake hands with the 5 who are already there, making a total of 15 shakes. When a seventh person arrives, she shakes hands with the 6 who are already there, making a total of 21 shakes.

Another way to solve this problem is to notice that each of the 7 people contributes 6 "half" (one hand) handshakes, for a total of 7 times 6 equals 42 "half" handshakes, or 21 full handshakes.

Yet another way is to draw a diagram, representing each person by a dot (vertex) and each handshake by a line (edge) connecting two vertices. Then all you have to do is count edges in the diagram! Notice that there are 7 outside edges (dark, thick lines), 7 edges forming a 7-pointed star (normal lines), and 7 dotted edges forming a fat star, making a total of 21.



5. Spiraling Squares Doodle. Ladybugs A and B start 2 inches apart. Since B is moving sideways to A's line of approach, the relative speed of A toward B is the same as if B were not moving at all. Thus, it takes A 20 seconds to reach B, and this happens when all 4 bugs reach the middle. Even though each bug is following a spiral path, we can figure out how long each spiral is by multiplying the bug's speed (1 inch per 10 seconds) by the time (20 seconds) to get the distance traveled (2 inches). Lo and behold—the spirals are the same length as the sides of the starting square!

6. The Disappearing Square. This paradox was invented by an amateur magician Paul Curry. If you actually make the pieces and rearrange them, you will find that the new "square" is no longer square, but it is now 7 and 1/7 inch tall. The "disappearing unit square" in the middle has somehow changed into a 7" by 1/7" rectangle (with the same area).

10.2 The Pony Engine's Puzzle. Jasper can do it in 18 moves as follows: he 1) comes out of his siding onto the bottom track; 2) moves to Abi and hooks up; 3) pulls Abi to bottom track; 4) stores her in the siding; 5) goes counterclockwise to hook up with Bruce; 6) pulls Bruce down and around to where Abi started, and then unhooks; 7) goes through the tunnel, around to the siding and hooks with Abi again; 8) pulls Abi out onto the bottom track; 9) pushes Abi to the left and up to contact Bruce, and hooks both cars together; 10) pulls both cars down to the bottom track; 11) pushes them both onto the siding, then unhooks Bruce; 12) pulls Abi over to where Bruce started, then unhooks her; 13) goes through the tunnel, and then counter-clockwise around back to the bottom track; 14) goes into the siding and hooks with Bruce; 15) pulls Bruce out of the siding, onto the bottom track; 16) pushes Bruce to the left and up to Abi's starting position, then unhooks; 17) chugs back to the bottom track; 18) reverses direction, chugs back onto the siding, and goes on home, exhausted!